

Symplectic structure of post-Newtonian Hamiltonian for spinning compact binaries

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The phase space of a Hamiltonian system is symplectic. However, the post-Newtonian Hamiltonian formulation of spinning compact binaries in existing publications does not have this property, when position, momentum and spin variables $[\mathbf{X}, \mathbf{P}, \mathbf{S}_1, \mathbf{S}_2]$ compose its phase space. This may give a convenient application of perturbation theory to the derivation of the post-Newtonian formulation, but also makes classic theories of a symplectic Hamiltonian system be a serious obstacle in application, especially in diagnosing integrability and nonintegrability from a dynamical system theory perspective. To completely understand the dynamical characteristic of the integrability or nonintegrability for the binary system, we construct a set of conjugate spin variables and reexpress the spin Hamiltonian part so as to make the complete Hamiltonian formulation symplectic. As a result, it is directly shown with the least number of independent isolating integrals that a conservative Hamiltonian compact binary system with both one spin and the pure orbital part to any post-Newtonian order is typically integrable and not chaotic. And a conservative binary system consisting of two spins restricted to the leading order spin-orbit interaction and the pure orbital part at all post-Newtonian orders is also integrable, independently on the mass ratio. For all other various spinning cases, the onset of chaos is possible.

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I. INTRODUCTION

Gravitational waves from coalescing spinning compact binaries, made of neutron stars and/or black holes, are important sources for ground-based and future space-borne detectors. Since a successful detection requires theoretical gravitational-wave templates matched with experimental data, the dynamics of two spinning compact bodies has recently been a hot issue in post-Newtonian (PN) celestial mechanics. As a current breakthrough in this field, there are several different methods for deriving the equations of motion of two point-like particles up to 2PN order [1,2] and 3PN order [3,4], and even up to higher-order PN approximations in general relativity [5]. One of these methods refers to the PN Lagrangian formulation, giving the orbits of black hole pairs in harmonic coordinates and in a general frame [4]. Another deals with the PN Hamiltonian formulation, describing the motion of two compact bodies in Arnowitt-Deser-Misner (ADM) coordinates and in the center-of-mass frame [3]. These two formulations have been proved to be approximately but not exactly equivalent [6,7]. It should be noted that the so-called physical *equivalence* between the two approaches is only based on a certain PN order accuracy, namely, has a small difference in the PN order approximation. For instance, conserved quantities of motion (if they exist) for the former are generally accurate to the PN order level, while they are rigorously invariant

for the latter.

However, there may be a great difference between the two formulations from a dynamical point of view. In the case of a comparable mass binary system with only one spinning body, the 2PN Hamiltonian dynamics shows no chaos due to the integrability of the system [8,9], but the 2PN Lagrangian dynamics was identified to be chaotic by the method of fractal basin boundaries built on the unstable, fractal set of periodic orbits [10,11]. These results are still correct when the two approaches give place to the 2PN Hamiltonian formulation of two equal-mass compact objects with two spins having the spin effects restricted to the leading order spin-orbit interaction [8,9] and its corresponding 2PN Lagrangian formulation [12], respectively. As a little attention to deserve, these distinct results seem to be explicit conflict if the difference between the two approaches is neglected. In this sense, it is natural to initially yield some doubt regarding the results. In fact the related arguments had never stopped until Levin [11] pointed out that these results are stemmed from different approximations to the same physical problem. Of course, the use of different indicators of chaos is also an important source leading to these arguments. At the beginning, the presence of chaos in the conservative 2PN Lagrangian dynamics of spinning compact binaries was confirmed by means of the fractal basin boundary method [13]. This implies that there are unpredictable gravitational waveforms during the inspiral. At once, the claim was questioned in Ref. [14] that calculates Lyapunov exponents as the divergence rate of nearby trajectories and finds no positive but zero Lyapunov exponents in all cases tested. The analysis on the

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chaos ruled out was strongly criticized in Refs. [12,15], where positive Lyapunov exponents are still obtained and the reason for the false Lyapunov exponents appeared in Ref. [14] is attributed to continually rescaling the shadow trajectory. In spite of this, we showed in a previous article [16] that no space-time coordinate redefinition ambiguity mentioned in [17] but a slightly different computational treatment of Lyapunov exponents is an exact source for these different results between Ref. [14] and Ref. [15]. The Lyapunov exponents in Ref. [14] are determined by the limit method for the computation of the stabilizing limit values as reliable values of Lyapunov exponents, while they are in Refs. [12,15] given by the fit method taking the slopes of the fit line about the natural logarithm of the divergence rate of nearby trajectories vs time as values of Lyapunov exponents. Clearly, the limit method becomes more difficult to detect chaos from order than the fit method when integration time is not long enough. Further, we argued the onset of the chaotic behavior of a pair of comparable mass black holes having one spin or two spins for the 2PN Lagrangian formulation by the invariant fast Lyapunov indicators of two nearby trajectories proposed in Ref. [18], viewed as a more sensitive tool to find chaos. Other reference [19] also supported this fact with aid of the frequency map analysis. As an exceptional case, the radial motion of spinning compact binaries in the Lagrangian formulation with contributions from the spins, mass quadrupole and magnetic dipole moments is explicitly integrated [20]. In addition, chaos in the conservative 2PN or 3PN Hamiltonian approach of compact binaries having two spins can be seen from the paper [21] of Hartl and Buonanno who adopted the fit method to calculate the Lyapunov exponents. As a point to emphasize, although both Ref. [10] and Ref. [21] admit the existence of chaos in the conservative 2PN Lagrangian or Hamiltonian dynamics of compact binaries with two spins, there are different opinions with respect to the dependence of chaos on dynamical parameters and initial conditions. For example, it was said in Ref. [10] that chaos becomes strong for the spins perpendicular to the orbital angular momentum, but it was reported in Ref. [21] that chaos is greatly possible to occur when initial spin vectors are nearly antialigned with the orbital angular momentum for the $(10+10)M_\odot$ configuration (M_\odot being mass of the Sun). The reason for the discrepancy was explained in our another work [22]. It was shown that no single physical parameter or initial condition but a complicated combination of all parameters and initial conditions affects the transition to chaos. As concluded in [22], one should distinguish these distinct results on chaos and order of spinning compact binaries in some references according to different approximations to this physical model, methods finding chaos, dynamical parameters and initial conditions.

It should be noticed that the above results (except those in Refs. [8,9]) associated to the dynamics of order and chaos are all from numerical investigations. In principle, numerical investigations, which do closely depend

on numerical integrators as well as indicators of chaos, dynamical parameters and initial conditions, are only a check of the local dynamics but not a check of the global dynamics. The so-called global structure of phase space scanned by the fractal basin boundary method [11] or the fast Lyapunov indicators [22] is still based on some specific dynamical parameters and initial conditions. In this case, it is regarded to as a partial but not thorough check. Additionally, although the method of finding parametric solutions to the Hamiltonian dynamics used in Refs. [8,9] is thought as an analytical method that can study the global dynamics, it has the limitation of application. In fact, a better and more rigorous method is to use the least number of independent constants of motion as a criterion for the prediction of the integrability or the nonintegrability hiding possible and potential chaos (the relationship between the integrability and the least number of independent constants will be introduced in Section II). Especially for these conservative Hamiltonian formulations of spinning compact binaries in which the constants of motion are exactly conserved, the method should work well without question. Unfortunately, a problem lies in that the phase space made of position, momentum and spin variables $[\mathbf{X}, \mathbf{P}, \mathbf{S}_1, \mathbf{S}_2]$ of these Hamiltonians is not completely symplectic in known literature [3, 23-27]. Although this plays an important role in providing the PN Hamiltonian formulation by the convenient application of perturbation theory, it has an obvious disadvantage that many Hamiltonian system properties have no way to be applied for these systems. For example, a closed nondegenerate differential 2-form, i.e., the so-called standard symplectic structure on a manifold [28] cannot be defined clearly. In particular, the relationship between the integrability and the least number of independent constants cannot be understood definitely. In view of the need of both a complete Hamiltonian theory and a dynamical system theory, the main motivation of the present paper is to design a group of new spin variables to rewrite the spin Hamiltonian part of the conservative PN Hamiltonian approximation for spinning compact binaries and to make the phase space of the whole Hamiltonian system have the symplectic structure so that we can apply the least number of independent constants to judge the integrability or the nonintegrability of the symplectic binary system and can further provide some theoretical insight into the global dynamics.

The rest of this paper is organized as follows. At first, we present a set of conjugate spin variables with the symplectic structure in Section II. Then, several advantages of using them are listed in Section III. Finally, Section IV summarizes the main results.

II. CONSTRUCTION OF CONJUGATE SPIN COORDINATES

In this section, let us introduce the conservative PN Hamiltonian formulation of spinning compact binaries,

where the pure orbital (nonspinning) part is accurate to 3PN order, and the spin part arrives at the 4PN order approximation. Meanwhile, the evolution equations of state variables and conserved quantities of motion in the system are given. In addition, a method finding conjugate spin variables with the symplectic structure is presented.

A. Conserved quantities in the PN Hamiltonian formulation of spinning compact binaries

The conservative Hamiltonian of spinning compact binaries is

$$H(\mathbf{X}, \mathbf{P}, \mathbf{S}_1, \mathbf{S}_2) = H_O(\mathbf{X}, \mathbf{P}) + H_S(\mathbf{X}, \mathbf{P}, \mathbf{S}_1, \mathbf{S}_2) \quad (1)$$

with the pure orbital part

$$H_O = H_{O,N} + H_{O,1PN} + H_{O,2PN} + H_{O,3PN} \quad (2)$$

and the spin part consisting of spin-orbit (SO) coupling, spin-spin (S^2) coupling and higher-order spin effects

$$H_S = H_{SO,1.5PN} + H_{SO,2.5PN} + H_{S^2,2PN} + H_{S^2P^2,3PN} + H_{S^3P,3.5PN} + H_{S^4,4PN}. \quad (3)$$

The notation S^2P^2 represents various possible coupling terms with respect to quadric forms of momentum \mathbf{P} and those of spins \mathbf{S}_1 and \mathbf{S}_2 . The notation S^3P refers to couplings between momentum \mathbf{P} and cubic terms of \mathbf{S}_1 and \mathbf{S}_2 , and S^4 stands for quartic terms of \mathbf{S}_1 and \mathbf{S}_2 . Ref. [24] has given H_O , $H_{SO,1.5PN}$ and $H_{S^2,2PN}$ in the center-of-mass frame. $H_{SO,2.5PN}$ can be found in [25], and $H_{S^4,4PN}$ is calculated in [26]. In addition, $H_{S^2P^2,3PN}$ and $H_{S^3P,3.5PN}$ are provided in [27]. Note that these Hamiltonians are directly given in the general frame, but they are easily changed into ones in the center-of-mass frame by means of the relation between the two frames given in [4]. Position \mathbf{X} and momentum \mathbf{P} are a set of canonical variables that satisfy the Hamiltonian equations of motion

$$\frac{d\mathbf{X}}{dt} = \frac{\partial H}{\partial \mathbf{P}}, \quad \frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{X}}. \quad (4)$$

The spin-evolution equations read [23]

$$\frac{d\mathbf{S}_i}{dt} = \frac{\partial H_S}{\partial \mathbf{S}_i} \times \mathbf{S}_i \quad (i = 1, 2). \quad (5)$$

Besides the total energy (1), five conserved quantities for the system are the constant magnitude of spin vectors

$$S_i^2 = (\chi_i m_i^2)^2 \quad (i = 1, 2), \quad (6)$$

and the total angular momentum [23]

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 \quad (7)$$

with the Newtonian-looking angular momentum $\mathbf{L} = \mathbf{X} \times \mathbf{P}$. Here dimensionless spin parameters $\chi_i \in [0, 1]$

are allowed for physically accessible realistic black hole or neutron star spins, and m_i denotes the mass of body i . In short, there are six independent constants or integrals of motion [41].

Now there is a problem whether the system (1) having the 6 integrals of motion in the 12-dimensional space made of $[\mathbf{X}, \mathbf{P}, \mathbf{S}_1, \mathbf{S}_2]$ is integrable. In order to answer it, let us recall the criterion of integrability of a Hamiltonian system. One must know $2n$ first integrals so as to obtain the analytical solutions of a system of $2n$ ordinary differential equations [28]. But it is often sufficient to know only n first integrals for a *canonical* system of differential equations, whose phase space is symplectic. Precisely speaking, the criterion of integrability is attributed to Liouville's theorem that an autonomous Hamiltonian with n degrees of freedom (i.e., with a $2n$ -dimensional phase space) is integrable if it has n independent integrals in involution [28]. Saying this in another way, a canonical Hamiltonian with n degrees of freedom is integrable if and only if there are n independent isolating integrals [29]. Strong Jeans theorem [30] implies that the n , as the required least number of independent isolating integrals for identifying the integrability, corresponds to the case of all regular with incommensurable frequencies. It can be inferred from this criterion that the number of isolating integrals for the integrability of the system (1) should be at least 9 rather than 6 due to the use of the spin-evolution equations (5) unlike the canonical equations (4), that is, the global phase space of the system (1) being nonsymplectic. Thus the existence of the above 6 integrals does not sufficiently show the integrability of the system (1). As mentioned in the Introduction, this is also checked numerically in the work [21]. To form this symplectic structure, we will construct new spin variables in place of the old ones.

B. A transformation to conjugate spin variables

Let the spin vectors be expressed in *cylindrical-like coordinates* $(\rho_i, \theta_i, \xi_i)$ as

$$\mathbf{S}_i = \chi_i m_i^2 \hat{\mathbf{S}}_i \quad (8)$$

with unit spin vectors

$$\hat{\mathbf{S}}_i = \begin{pmatrix} \rho_i \cos \theta_i \\ \rho_i \sin \theta_i \\ k_i \xi_i \end{pmatrix}, \quad (9)$$

where each ρ_i depends on ξ_i as follows

$$\rho_i = \sqrt{1 - (k_i \xi_i)^2}. \quad (10)$$

In the above equation, two coefficients k_i are what we shall determine. In fact, Eq. (8) gives a transformation from the old spin variables to the new ones in the form

$$\mathbf{S}_i : (S_{i1}, S_{i2}, S_{i3}) \rightarrow (\theta_i, \xi_i), \quad (11)$$

where subscript j denotes the j th-component S_{ij} of the spin vector \mathbf{S}_i . That is to say, each spin containing 3 Cartesian spin components is a function of the 2 new spin variables, marked as $\mathbf{S}_i = \mathbf{S}_i(\theta_i, \xi_i)$.

Using the new spin variables $[\theta_1, \theta_2, \xi_1, \xi_2]$, namely inserting Eq. (8) into Eq. (3), we rewrite H_S as

$$\mathbb{H}_S(\mathbf{X}, \boldsymbol{\theta}; \mathbf{P}, \boldsymbol{\xi}) = H_S[\mathbf{X}, \mathbf{P}, \mathbf{S}_1(\theta_1, \xi_1), \mathbf{S}_2(\theta_2, \xi_2)] \quad (12)$$

with $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Suppose that (θ_i, ξ_i) are canonical coordinates in the phase space with symplectic structure, and then we have canonical spin Hamiltonian equations

$$\frac{d\theta_i}{dt} = \frac{\partial \mathbb{H}_S}{\partial \xi_i}, \quad \frac{d\xi_i}{dt} = -\frac{\partial \mathbb{H}_S}{\partial \theta_i}. \quad (13)$$

It is clear that this hypothesis is true if Eq. (5) is equivalent to Eq. (13). The details of derivation are described in the following.

It is easy to obtain

$$\begin{aligned} \frac{\partial \mathbb{H}_S}{\partial \xi_i} &= \sum_{j=1}^3 \frac{\partial H_S}{\partial S_{ij}} \frac{\partial S_{ij}}{\partial \xi_i} = k_i \chi_i m_i^2 \left[-\frac{k_i \xi_i}{\rho_i} \right. \\ &\quad \times \left(\frac{\partial H_S}{\partial S_{i1}} \cos \theta_i + \frac{\partial H_S}{\partial S_{i2}} \sin \theta_i \right) + \left. \frac{\partial H_S}{\partial S_{i3}} \right], \\ \frac{\partial \mathbb{H}_S}{\partial \theta_i} &= \sum_{j=1}^2 \frac{\partial H_S}{\partial S_{ij}} \frac{\partial S_{ij}}{\partial \theta_i} \\ &= -\chi_i m_i^2 \rho_i \left(\frac{\partial H_S}{\partial S_{i1}} \sin \theta_i - \frac{\partial H_S}{\partial S_{i2}} \cos \theta_i \right). \end{aligned}$$

According to the transformation (8), we have

$$\begin{aligned} \frac{d\mathbf{S}_i}{dt} &= \chi_i m_i^2 \frac{d\hat{\mathbf{S}}_i}{dt} \\ &= \chi_i m_i^2 \begin{pmatrix} -\frac{k_i^2 \xi_i}{\rho_i} \cos \theta_i \frac{d\xi_i}{dt} - \rho_i \sin \theta_i \frac{d\theta_i}{dt} \\ -\frac{k_i^2 \xi_i}{\rho_i} \sin \theta_i \frac{d\xi_i}{dt} + \rho_i \cos \theta_i \frac{d\theta_i}{dt} \\ k_i \frac{d\xi_i}{dt} \end{pmatrix} \\ &= \chi_i m_i^2 \begin{pmatrix} \frac{k_i^2 \xi_i}{\rho_i} \cos \theta_i \frac{\partial \mathbb{H}_S}{\partial \theta_i} - \rho_i \sin \theta_i \frac{\partial \mathbb{H}_S}{\partial \xi_i} \\ \frac{k_i^2 \xi_i}{\rho_i} \sin \theta_i \frac{\partial \mathbb{H}_S}{\partial \theta_i} + \rho_i \cos \theta_i \frac{\partial \mathbb{H}_S}{\partial \xi_i} \\ -k_i \frac{\partial \mathbb{H}_S}{\partial \theta_i} \end{pmatrix} \\ &= k_i (\chi_i m_i^2)^2 \begin{pmatrix} k_i \xi_i \frac{\partial H_S}{\partial S_{i2}} - \rho_i \sin \theta_i \frac{\partial H_S}{\partial S_{i3}} \\ -k_i \xi_i \frac{\partial H_S}{\partial S_{i1}} + \rho_i \cos \theta_i \frac{\partial H_S}{\partial S_{i3}} \\ \rho_i \left(\frac{\partial H_S}{\partial S_{i1}} \sin \theta_i - \frac{\partial H_S}{\partial S_{i2}} \cos \theta_i \right) \end{pmatrix} \\ &= k_i \chi_i m_i^2 \begin{pmatrix} S_{i3} \frac{\partial H_S}{\partial S_{i2}} - S_{i2} \frac{\partial H_S}{\partial S_{i3}} \\ -S_{i3} \frac{\partial H_S}{\partial S_{i1}} + S_{i1} \frac{\partial H_S}{\partial S_{i3}} \\ S_{i2} \frac{\partial H_S}{\partial S_{i1}} - S_{i1} \frac{\partial H_S}{\partial S_{i2}} \end{pmatrix} \\ &= k_i \chi_i m_i^2 \frac{\partial H_S}{\partial \mathbf{S}_i} \times \mathbf{S}_i. \end{aligned}$$

If we take $k_i = 1/(\chi_i m_i^2)$, the above equation just agrees with Eq. (5). Inversely, in the similar way we can also derive Eq. (13) from Eq. (5). Therefore, conjugate spin variables (θ_i, ξ_i) whose time evolutions are given by Eq.

(13) are what we want. The system (1) and the precession equations (5) can be reexpressed as a new complete canonical formalism

$$\mathbb{H}(\mathbf{X}, \boldsymbol{\theta}; \mathbf{P}, \boldsymbol{\xi}) = H[\mathbf{X}, \mathbf{P}, \mathbf{S}_1(\theta_1, \xi_1), \mathbf{S}_2(\theta_2, \xi_2)] \quad (14)$$

and the precession equations (13), respectively. This means that there are only two independent new spin variables in the spin precession equations (13) per compact body. Here we specify $\chi_i \neq 0$. If one of χ_1 and χ_2 vanishes, the other nonzero spin vector needs rewriting in the form (8). If $\chi_1 = \chi_2 = 0$, the pure orbital part itself is of the canonical formalism. We also find that it is impossible to get conjugate spin variables if the original spin vectors \mathbf{S}_i are expressed in *spherical coordinates*.

It should again be emphasized that the definition of the word ‘‘canonical’’ mentioned above does completely coincide with one given by the book entitled Classical Mechanics [31]. In other words, two components θ_i and ξ_i of each spin are said to be canonical or conjugate variables if their time evolutions can satisfy Eq. (13). In this case, the phase space of the system (14) is completely symplectic. In a word, the canonical or conjugate variables we called in this paper can equip the phase space of the system (14) with a complete symplectic structure. As an important illustration, the term ‘‘canonical spin’’ appeared in some references [32-34] means using canonical Dirac brackets instead of the Poisson brackets when the equations of motion are derived from that Hamiltonian. An explicit difference between their spin variables and ours lies in that the former appears as a spin tensor, while the latter relates to a two-dimensional vector. Of course, the spin tensor can also be defined as a three-dimensional spin vector like Eq. (4.26) of Ref. [33]. Still the spin-evolution equations do resemble Eq. (5) rather than Eq. (13). The facts have shown clearly that the meaning of the canonical in these articles is not consistent with ours.

III. ADVANTAGES OF USING THE NEW SPIN VARIABLES

It can easily be observed that the expression of \mathbb{H} is more complicated than that of H . This may explain why known references use the old spin variables rather than the new ones to derive the Hamiltonian formulations. Nevertheless, the use of the new spin variables has more advantages from Hamiltonian dynamics. We list following several main points.

(i) Reduction of dimensionality. The use of the new spin variables automatically satisfies the two constraints by Eq. (6) such that a problem of 12-dimensional space is reduced to one of 10-dimensional phase space. That is to say, the 12 components of $[\mathbf{X}, \mathbf{P}, \mathbf{S}_1, \mathbf{S}_2]$ in Eq. (1) are changed into the 10 components of $[\mathbf{X}, \boldsymbol{\theta}; \mathbf{P}, \boldsymbol{\xi}]$ in Eq. (14). The reduction of two variables can also be seen from the transformation (11). It should be noted particularly that the information on the constancy of the

magnitude of the old spin vectors does hide in the new variables. In fact, for any time the new variables are always constrained by unit spin vectors (9) when $\theta_i \in [0, 2\pi]$ and $\xi_i \in [-\chi_i m_i^2, \chi_i m_i^2]$.

(ii) Symplectic geometry. The complete canonical formalism (14) contains the symplectic structure expressed as

$$\omega^2 = \sum_{j=1}^3 dX_j \wedge dP_j + \sum_{i=1}^2 d\theta_i \wedge d\xi_i. \quad (15)$$

Then the volume form of the 10-dimensional phase space is defined by the local coordinate representation

$$\omega^{10} = \Pi_{j=1}^3 dX_j \wedge dP_j \wedge \Pi_{i=1}^2 d\theta_i \wedge d\xi_i. \quad (16)$$

The symbol \wedge means wedge product. Meanwhile the integral invariants of Poincaré exist. In short, the related theories of symplectic geometry [28] are fully suitable for the completely canonical Hamiltonian system, \mathbb{H} .

(iii) Symplectic integration algorithms. A class of important numerical schemes, called symplectic integrators [35–37], can be used to preserve both the accuracy of essential properties and the symplectic structure of the canonical system \mathbb{H} . Unfortunately, the system is difficultly separated into two integrable pieces such that explicit symplectic integrators become useless. In spite of this, implicit symplectic methods such as the implicit midpoint method [36] are always efficient.

(iv) Dynamics of compact binaries with one spinning body. A conservative Hamiltonian binary system is an 8-dimensional dynamical problem with the symplectic structure if only one body spins. This symplectic system holds four constants of motion including the total energy and the total angular momentum. This sufficiently shows the integrability of the symplectic system, *regardless of* the PN order. Orbits are confined to a 4-dimensional torus. This is an extension to the result of Refs. [8,9] that there is no chaos in the 2PN Hamiltonian formulation of two compact objects when one body spins.

Perhaps someone casts doubt on the convincing of the result that the conservative symplectic Hamiltonian for one spinning body at any PN order to either the pure orbital part or the spin part is completely integrable because the symplectic Hamiltonian is not known at all PN orders. It just demonstrates the superior properties of the symplectic Hamiltonian system. We emphasize again that the result is always correct *only if* the future higher-order PN approximations are given in ensuring the existence of these four integrals including the total energy and the total angular momentum. Even the result ought to hold in the extreme mass ratio limit at all PN orders. This seems to have an apparent contradiction with Suzuki & Maeda's result that the dynamics is chaotic at least for unphysical large values of the spin [38]. In spite of the limit case, the dynamical model considered in this paper and the system consisting of a Schwarzschild black hole and a spinning test particle in Ref. [38] are still not equivalent. Four typical differences

between them are listed here. (a) Mechanisms of dynamical approaches. The former is from PN approximations, while the latter using the Papapetrou-Dixon equations of motion is fully relativistic. (b) Symplectic structure. The former belongs to a symplectic Hamiltonian system, but the latter does not. (c) Dimensions of variables. For the former the position, momentum and spin vectors are 3, 3 and 2 dimensions, respectively, while each of the position, momentum and spin vectors has 4 dimensions for the latter. (d) Number of integrals. In particular, the difference between them becomes clearer by counting the number of their integrals. For the former the four integrals are always present in the 8-dimensional phase space with the symplectic structure, but for the latter one finds the presence of five constraints including the relation for the mass of the particle, the constant magnitude of the spin vector, the spin supplementary condition, the energy of the particle and the z component of the total angular momentum. As stated above, the five constraints are much less than the required least number of independent isolating integrals for the integrability of the 12-dimensional nonsymplectic system. This fact is supported by the result of Suzuki & Maeda. Recently, Ref. [39] also found chaotic orbits for smaller spin values in the motion of spinning test particles around a Schwarzschild field. Therefore, it can be concluded that there is actually no explicit conflict between our result and Suzuki & Maeda's result. In addition, as far as the corresponding PN Lagrangian formulation associated with the PN Hamiltonian of one spinning body is concerned, the so-called constants except the constant magnitude of the spin are approximately conserved at a certain PN order, that is, the desired least number of integrals for the integrability is not rigorously satisfied, so it is no surprise to see the onset of chaos [10,11].

(v) Dynamics of compact binaries having two spins. The four conserved quantities involving the total energy and the total angular momentum (7) do not sufficiently show that the symplectic system \mathbb{H} in the 10-dimensional phase space is integrable. In fact, a fifth integral of motion is absent. Thus the symplectic system is nonintegrable. From this point of view, it is easy to understand the result of [21] that the 2PN Hamiltonian binary system consisting of the leading order spin-orbit integration and the spin-spin coupling is chaotic for some specific dynamical parameters and initial conditions. On the other hand, a conservative symplectic Hamiltonian spinning binary system containing both the pure orbital part at all PN orders and the leading order spin-orbit coupling has two additional conserved quantities [23]:

$$\mathbf{L} \cdot \mathbf{L} = \text{const}, \quad \mathbf{L} \cdot \mathbf{S}_{eff} = \text{const} \quad (17)$$

with $\mathbf{S}_{eff} = [2 + 3m_2/(2m_1)]\mathbf{S}_1 + [2 + 3m_1/(2m_2)]\mathbf{S}_2$. In this sense, the total energy and these five independent constants given by Eqs. (7) and (17) sufficiently determine the integrability of the 10-dimensional symplectic system. This is another extension to the result of [8,9] that there is no chaos in the 2PN Hamiltonian

formulation of two compact objects with two spins when the binaries are of equal mass and spin effects are limited to the leading order spin-orbit couplings only. Here are two points to emphasize. First, because the integrability needs only five independent constants, these six independent constants show that two of five frequencies are commensurable. In this case, a resonance may occur. Second, it is not necessary to use the constraint of *equal mass* demanded in Refs. [8,9], since the existence of these six independent constants does not depend on any mass. If this constraint is considered, there seem to be two new additional constants

$$\mathbf{S} \cdot \mathbf{S} = \text{const}, \quad \mathbf{L} \cdot \mathbf{S} = \text{const} \quad (18)$$

with $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. But it should be noted that there are two relations among Eqs. (7), (17) and (18), $2\mathbf{L} \cdot \mathbf{S}_{\text{eff}} = 7\mathbf{L} \cdot \mathbf{S}$ and $\mathbf{J} \cdot \mathbf{S} = \mathbf{L} \cdot \mathbf{S} + \mathbf{S}^2$. Consequently, no new conserved quantity appears.

Obviously, the use of the complete symplectic formalism (14) is rigorous enough to show that the conservative symplectic Hamiltonian formulations for the two cases of spin effects we discussed above are completely regular. It makes our procedure greatly superior to the method of finding parametric solutions to the Hamiltonian dynamics [8,9] and the technique of fractal basin boundaries used in [11]. If various higher-order PN expansions are considered, it is not easy to obtain parametric solutions, and the fractal method, as a numerical tool for detecting chaos from order, is a check of the dynamics only in certain situations but not a full proof of integrability, or nonintegrability. As stated in [40], “The only definite proof of integrability is by finding the analytic forms of the integrals”. Our treatment is just fit for this requirement. In addition, it is worth noting that both the dynamics of the Hamiltonian formulation and one of the Lagrangian formulation are completely different in these two spin effects.

IV. CONCLUSIONS

In the PN Hamiltonian formulation of spinning compact binaries, we have designed a group of conjugate

spin variables that satisfy the canonical spin-evolution equations. This treatment makes the complete Hamiltonian formulation symplectic. Seen from Hamiltonian mechanics and a dynamical system theory, the construction of the symplectic Hamiltonian formulation is so important that all properties of Hamiltonian dynamical systems can directly be applied to the symplectic system. The obtained symplectic Hamiltonian formulation bring the above-mentioned advantages. Above all, one of them is that the least number of independent isolating integrals (equivalent to the half number of all conjugate state variables) can be regarded to as a criterion for the prediction of the integrability or the nonintegrability of the symplectic Hamiltonian binary system. As a result, it is strictly shown through some theoretical insight that the conservative symplectic PN Hamiltonian dynamics of spinning compact binaries is integrable and nonchaotic for the two above-mentioned cases of one spinning body and the leading spin-orbit interaction.

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- [41] Strictly speaking, a constant of motion and an integral of motion are two different concepts. An apparent difference between them lies in that the former is a function of coordinates, velocities (or momenta) and time, while the latter is a function of the phase space coordinates. Every integral is a constant of the motion, but the converse may not be true. Here the constant considered is only a function of coordinates and momenta, so they are both the same.